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Conservation laws for NQC-type difference equations

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Abstract
This paper presents a classification of all three-point conservation laws for a large class of integrable difference equations that has been described by Nijhoff, Quispel and Capel. We show that every nonlinear equation from this class has at least two nontrivial conservation laws. Most of the conservation laws that are found are new.

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1. Introduction

Conservation laws play an important role in the theory of partial differential equations (PDEs). Some conservation laws express physical quantities; even those that do not are usually of mathematical interest. In particular, one definition of integrability is the possession of infinitely many higher symmetries or conservation laws [5, 15]. There are two main ways to calculate conservation laws for a given PDE. If the PDE has a variational formulation then Noether's Theorem yields a conservation law corresponding to each generator of Lie symmetries of the variational problem. Noether's Theorem has been adapted to deal also with PDEs that have a Hamiltonian or multisymplectic structure. Alternatively, conservation laws of all types of PDEs may be found by a direct method which does not use symmetries [2, 19].

The theory of conservation laws for partial difference equations (PΔEs) mirrors that for PDEs. Hydon and Mansfield [10] have formulated the basic theory for the realization of conservation laws in a discrete space. Just as for PDEs, the discrete analogue of Noether's theorem [4, 11, 13, 23] is applicable only to equations with a known variational, Hamiltonian or multisymplectic structure. Until recently, this condition greatly restricted the class of PΔEs for which conservation laws could be found. This is partly because symmetry analysis of difference equations was not introduced until Maeda's 1987 paper [14], and it has since been developed along several different lines [8, 12, 20]. Furthermore, symmetry calculations are typically lengthy, and only a few of the symmetries that have been found so far are variational symmetries.

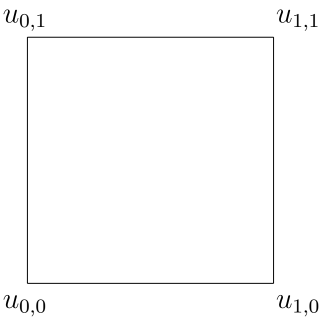

Figure 1. The values of the dependent variable at the vertices of a quad-graph.

However, a discrete analogue of the direct method has now been developed by Hydon [9]; this was used to find all three-point conservation laws for the discrete potential modified Korteweg–de Vries (dpmKdV) equation. An improvement to Hydon's technique was introduced in [22] and was used to find all three-point and five-point conservation laws of the discrete Korteweg–de Vries (dKdV) equation.

At present, there are various criteria for P∆Es to be integrable [1, 3, 16, 17]. Attention has largely focused on quad-graphs (figure 1). These have a scalar dependent variable u that is defined on the domain $\mathbb{Z}^2$; we shall use the coordinates (k, l) as the independent variables. For brevity we denote the values of u at the vertices of the quad-graph by $u_{0,0} = u(k, l), u_{1,0} = u(k+1, l), u_{0,1} = u(k, l+1), u_{1,1} = u(k+1, l+1)$, as shown.

The easiest definition of integrability for quad-graphs is consistency on a cube. This implies the existence of a Lax pair [3], so that the system is integrable via a spectral problem. Adler, Bobenko and Suris have classified all integrable quad-graphs that are consistent on a cube, linear in each variable, invariant under the symmetries of a square, and possess the tetrahedron property [1]. Hietarinta discovered a quad-graph that lacks the tetrahedron property but has a Lax pair [7]. However, this has recently been shown to be linearizable [21], so it can be integrated without recourse to the spectral problem. It is not yet known whether all quad-graphs that have a Lax pair but lack the tetrahedron property can be linearized.

Classification does not tell us whether a given quad-graph (that does not satisfy the assumptions of any known classification) is integrable. Various tests have been developed which indicate (but do not prove) integrability; the most notable of these is the method of singularity confinement [6]. Another possibility is to seek symmetries or conservation laws. We are beginning to develop this approach by investigating the conservation laws of equations that are known to be integrable. Except for the dKdV and dpmKdV equations, little is known about the conservation laws of quad-graphs. The purpose of the current paper is to use the direct method [9, 22] to classify the three-point conservation laws of a wide class of integrable quad-graphs that were introduced by Nijhoff, Quispel and Capel [18]. These are of the form

$$\frac{(p+s)u_{1,0} - (p-r)u_{0,0} - 1}{(q+s)u_{0,1} - (q-r)u_{0,0} - 1} = \frac{(q+r)u_{1,1} - (q-s)u_{1,0} - 1}{(p+r)u_{1,1} - (p-s)u_{0,1} - 1}, \tag{1}$$

where p, q, r, s are constants. We shall refer to (1) as the NQC equation; the factors $(p \pm r)$, $(p \pm s)$, $(q \pm r)$ and $(q \pm s)$ will be called the *coefficients* of the NQC equation.

The structure of this paper is as follows: section 2 describes the direct method for calculating conservation laws for quad-graph equations. The NQC equation is split into equivalence classes in section 3; this reduces the problem of finding the conservation laws to the analysis of four cases. The main theorem that classifies all three-point conservation laws for these cases is stated in section 4; an example in section 5 demonstrates how to extend the classification to any nondegenerate choice of coefficients. We conclude in section 6 with a brief discussion of some open problems.

2. How to find three-point conservation laws for a given quad-graph

The general form of a scalar PΔE on the quad-graph is

$$P(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0. \tag{2}$$

A conservation law for any quad-graph equation (2) is an expression of the form

$$(S_k - \mathrm{id})F + (S_l - \mathrm{id})G = 0 \tag{3}$$

that is satisfied by all solutions of the equation. Here the functions F and G are the components of the conservation law, id is the identity mapping, and S_k and S_l are forward shifts of the coordinates k and l respectively:

$$S_k : (k, l) \to (k + 1, l) \quad \text{and} \quad S_l : (k, l) \to (k, l + 1).$$

The shift operators induce the following mappings on the dependent variables:

$$S_k u(k, l) = u(k + 1, l), \qquad S_l u(k, l) = u(k, l + 1). \tag{4}$$

A conservation law is trivial if it holds identically (not just on solutions of the PΔE), or if F and G both vanish on all solutions of (2). We seek nontrivial conservation laws that lie on the quad-graph, so that (3) involves only $k, l, u_{0,0}, u_{1,0}, u_{0,1}$ and $u_{1,1}$. Consequently the functions F and G are of the form

$$F = F(k, l, u_{0,0}, u_{0,1}), \tag{5}$$

$$G = G(k, l, u_{0,0}, u_{1,0}). \tag{6}$$

As F and G depend upon three of the four points of the quad-graph, these are called *three-point conservation laws*. Higher-order conservation laws may also be sought by dropping the restriction that the conservation law lies on a single quad-graph.

If a nontrivial three-point conservation law has one component equal to zero then we can integrate the corresponding equation directly. For instance, if $F = 0$ then

$$(S_l - \mathrm{id})G(k, l, u_{0,0}, u_{1,0}) = 0,$$

which can be integrated once to yield the ordinary difference equation (OΔE)

$$G(k, l, u_{0,0}, u_{1,0}) = A(k), \tag{7}$$

where $A(k)$ is an arbitrary function. If a quad-graph equation has one conservation law of this type, it has infinitely many, because any function of $G(k, l, u_{0,0}, u_{1,0})$ is in the kernel of $(S_l - \mathrm{id})$. Such equations, together with linear equations, do not require a spectral method. Therefore we shall focus on nonlinear quad-graph equations with finitely many three-point conservation laws.

Now we explain the technique for calculating the conservation laws of the quad-graph equation (2). The three-point conservation laws can be determined directly by substituting (2) into

$$F(k + 1, l, u_{1,0}, u_{1,1}) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l + 1, u_{0,1}, u_{1,1}) - G(k, l, u_{0,0}, u_{1,0}) = 0, \tag{8}$$

and solving the resulting functional equation. Suppose that (2) can be solved for $u_{1,1}$ as follows:

$$u_{1,1} = \omega(k, l, u_{0,0}, u_{1,0}, u_{0,1}).$$

Then (8) amounts to

$$F(k+1, l, u_{1,0}, \omega) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, \omega) - G(k, l, u_{0,0}, u_{1,0}) = 0. \qquad (9)$$

In order to solve this functional equation we have to reduce it to a system of partial differential equations. To do this, first eliminate terms that contain ω by applying each of the following (commuting) differential operators to (9):

$$L_1 = \frac{\partial}{\partial u_{0,1}} - \frac{\omega_{u_{0,1}}}{\omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}}, \qquad L_2 = \frac{\partial}{\partial u_{1,0}} - \frac{\omega_{u_{1,0}}}{\omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}},$$

where $\omega_{u_{i,j}}$ denotes $\frac{\partial \omega}{\partial u_{i,j}}$. The operators L_1, L_2 differentiate with respect to $u_{1,0}, u_{0,1}$ respectively, keeping ω fixed. This procedure does not depend upon the form of ω; it can be applied equally to any quad-graph equation. After using operators L_1 and L_2 on equation (9) we have

$$L_1 L_2(F(k, l, u_{0,0}, u_{0,1}) + G(k, l, u_{0,0}, u_{1,0})) = 0. \qquad (10)$$

If (10) is divided by the factor that multiplies a particular derivative of $G(k, l, u_{0,0}, u_{1,0})$ and is then differentiated with respect to $u_{0,1}$, we obtain a functional differential equation which is independent of that derivative. This process is repeated for each derivative of $G(k, l, u_{0,0}, u_{1,0})$ and finally for $G(k, l, u_{0,0}, u_{1,0})$ itself; this produces a PDE for $F(k, l, u_{0,0}, u_{0,1})$. If the coefficients involve $u_{1,0}$, the PDE can be split into a system of PDEs.

Having differentiated the determining equation for a conservation law several times, we have created a hierarchy of functional differential equations that every three-point conservation law must satisfy. The functions F and G can be determined completely by going up the hierarchy, a step at a time, to determine the constraints that these equations place on the unknown functions. As the constraints are solved sequentially, more and more information is gained about the functions. At the highest stage, the determining equation (9) is satisfied, and the only remaining unknowns are the constants that multiply each conservation law. This is a straightforward but very lengthy process; for brevity, we shall omit the details of these calculations in our analysis of the NQC equation.

3. Simplification of the NQC equation

We begin by showing that the NQC equation,

$$\frac{(p+s)u_{1,0} - (p-r)u_{0,0} - 1}{(q+s)u_{0,1} - (q-r)u_{0,0} - 1} = \frac{(q+r)u_{1,1} - (q-s)u_{1,0} - 1}{(p+r)u_{1,1} - (p-s)u_{0,1} - 1}, \qquad (11)$$

is mapped to another NQC equation under the group of equivalence transformations D_4 generated by rotations and reflections.

First consider rotations. Let

$$\hat{k} = -l, \qquad \hat{l} = k, \qquad \hat{u}(\hat{k}, \hat{l}) = u(\hat{l}, -\hat{k}) = u(k, l).$$

Then

$$\hat{u}(\hat{k}-1, \hat{l}) = u(\hat{l}, 1-\hat{k}) = u(k, l+1),$$
$$\hat{u}(\hat{k}-1, \hat{l}+1) = u(\hat{l}+1, 1-\hat{k}) = u(k+1, l+1),$$
$$\hat{u}(\hat{k}, \hat{l}+1) = u(\hat{l}+1, -\hat{k}) = u(k+1, l).$$

Note that $S_{\hat{k}}^{-1}\hat{u}(\hat{k},\hat{l}) = S_l u(k,l)$ and $S_{\hat{l}}\hat{u}(\hat{k},\hat{l}) = S_k u(k,l)$. The NQC equation (11) can be rewritten as

$$\frac{(p+s)\hat{u}_{0,1} - (p-r)\hat{u}_{0,0} - 1}{(q+s)\hat{u}_{-1,0} - (q-r)\hat{u}_{0,0} - 1} = \frac{(q+r)\hat{u}_{-1,1} - (q-s)\hat{u}_{0,1} - 1}{(p+r)u_{-1,1} - (p-s)\hat{u}_{-1,0} - 1} \tag{12}$$

where $\hat{u}_{i,j} = \hat{u}(\hat{k}+i, \hat{l}+j)$. Apply $S_{\hat{k}}$ to (12) and then rearrange to obtain

$$\frac{(-q+r)\hat{u}_{1,0} - (-q-s)\hat{u}_{0,0} - 1}{(p+r)\hat{u}_{0,1} - (p-s)\hat{u}_{0,0} - 1} = \frac{(p+s)\hat{u}_{1,1} - (p-r)\hat{u}_{1,0} - 1}{(-q+s)\hat{u}_{1,1} - (-q-r)\hat{u}_{0,1} - 1}.$$

This is of the form

$$\frac{(\hat{p}+\hat{s})\hat{u}_{1,0} - (\hat{p}-\hat{r})\hat{u}_{0,0} - 1}{(\hat{q}+\hat{s})\hat{u}_{0,1} - (\hat{q}-\hat{r})\hat{u}_{0,0} - 1} = \frac{(\hat{q}+\hat{r})\hat{u}_{1,1} - (\hat{q}-\hat{s})\hat{u}_{1,0} - 1}{(\hat{p}+\hat{r})\hat{u}_{1,1} - (\hat{p}-\hat{s})\hat{u}_{0,1} - 1} \tag{13}$$

where

$$(\hat{p}, \hat{q}, \hat{r}, \hat{s}) = (-q, p, s, r). \tag{14}$$

Therefore this rotation generates the equivalence transformation

$$\Gamma_a\colon (p, q, r, s) \mapsto (-q, p, s, r). \tag{15}$$

Suppose that for a particular choice of parameters $(\hat{p}, \hat{q}, \hat{r}, \hat{s})$ equation (13) has a conservation law

$$(S_{\hat{k}} - \mathrm{id})\hat{F}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{0,1}) + (S_{\hat{l}} - \mathrm{id})\hat{G}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{1,0}) = 0. \tag{16}$$

Then, applying $S_{\hat{k}}^{-1}$ to this and expanding, we obtain

$$\hat{F}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{0,1}) - \hat{F}(\hat{k}-1, l, \hat{u}_{-1,0}, \hat{u}_{-1,1}) + \hat{G}(\hat{k}-1, l+1, \hat{u}_{-1,1}, \hat{u}_{0,1}) \\ - \hat{G}(\hat{k}-1, \hat{l}, \hat{u}_{-1,0}, \hat{u}_{0,0}) = 0.$$

In terms of the original variables, this amounts to

$$\hat{F}(-l, k, u_{0,0}, u_{1,0}) - \hat{F}(-l-1, k, u_{0,1}, u_{1,1}) + \hat{G}(-l-1, k+1, u_{1,1}, u_{1,0}) \\ - \hat{G}(-l-1, k, u_{0,1}, u_{0,0}) = 0,$$

which can be written as

$$(S_k - \mathrm{id})\hat{G}(-l-1, k, u_{0,1}, u_{0,0}) + (S_l - \mathrm{id})[-\hat{F}(-l, k, u_{0,0}, u_{1,0})] = 0. \tag{17}$$

This is a conservation law for the original NQC equation (11) with the parameters $(p, q, r, s) = (\hat{q}, -\hat{p}, \hat{s}, \hat{r})$.

In the same way, we can examine the effect of reflections upon conservation laws. Let

$$\hat{k} = -k, \qquad \hat{l} = l, \qquad \hat{u}(\hat{k}, \hat{l}) = u(-\hat{k}, \hat{l}) = u(k, l).$$

Then

$$\hat{u}(\hat{k}-1, \hat{l}) = u(1-\hat{k}, \hat{l}) = u(k+1, l),$$
$$\hat{u}(\hat{k}-1, \hat{l}+1) = u(1-\hat{k}, \hat{l}+1) = u(k+1, l+1),$$
$$\hat{u}(\hat{k}, \hat{l}+1) = u(-\hat{k}, \hat{l}+1) = u(k, l+1).$$

So $S_{\hat{k}}^{-1}\hat{u}(\hat{k},\hat{l}) = S_k u(k,l)$ and $S_{\hat{l}}\hat{u}(\hat{k},\hat{l}) = S_l u(k,l)$. As before, write the NQC equation (11) in terms of the new variable as follows:

$$\frac{(p+s)\hat{u}_{-1,0} - (p-r)\hat{u}_{0,0} - 1}{(q+s)\hat{u}_{0,1} - (q-r)\hat{u}_{0,0} - 1} = \frac{(q+r)\hat{u}_{-1,1} - (q-s)\hat{u}_{-1,0} - 1}{(p+r)\hat{u}_{-1,1} - (p-s)\hat{u}_{0,1} - 1}.$$

Now apply $S_{\hat{k}}$ and rearrange to obtain

$$\frac{(-p+r)\hat{u}_{1,0} - (-p-s)\hat{u}_{0,0} - 1}{(q+r)\hat{u}_{0,1} - (q-s)\hat{u}_{0,0} - 1} = \frac{(q+s)\hat{u}_{1,1} - (q-r)\hat{u}_{1,0} - 1}{(-p+s)\hat{u}_{1,1} - (-p-r)\hat{u}_{0,1} - 1}. \tag{18}$$

This is of the form (13) with

$$(\hat{p}, \hat{q}, \hat{r}, \hat{s}) = (-p, q, s, r). \tag{19}$$

Therefore the above reflection amounts to the equivalence transformation

$$\Gamma_b : (p, q, r, s) \mapsto (-p, q, s, r). \tag{20}$$

The conservation law (16) amounts (after applying $S_{\hat{k}^{-1}}$) to

$$(S_k - \mathrm{id})[-\hat{F}(-k, l, u_{0,0}, u_{0,1})] + (S_l - \mathrm{id})\hat{G}(-k-1, l, u_{1,0}, u_{0,0}) = 0. \tag{21}$$

Note that $r + s$ and $|p^2 - q^2|$ are invariants of the group generated by Γ_a and Γ_b.

For certain parameter values, the NQC equation is degenerate in one of two senses: either it can be factorized into a pair of linear PDEs, or it can be integrated to yield an OΔE. These degenerate cases are classified by the following two lemmas.

Lemma 1. *If $p^2 = q^2$ then the NQC equation is factorizable.*

Proof. Substituting $p = q$ into (11) and rearranging this as a polynomial equation gives the factorization

$$(u_{1,0} - u_{0,1})[(q + r)(q + s)u_{1,1} - (q - r)(q - s)u_{0,0} - 2q] = 0.$$

Substituting $p = -q$ into (11) gives

$$(u_{0,0} - u_{1,1})[(q + r)(q + s)u_{0,1} - (q - r)(q - s)u_{1,0} - 2q] = 0. \qquad \square$$

Lemma 2. *If at least one of p, q and at least one of r, s are zero then the NQC equation may be reduced to an OΔE.*

Proof. Without loss of generality, we may restrict attention to the case $q = r = 0$. All other cases may be obtained from this one by using Γ_a and Γ_b. Note that if $p = q = r = 0$ then (11) is a trivial equation, so assume that $p \neq 0$.

If $s = 0$ then the NQC equation reduces to

$$[1 + p(u_{0,1} - u_{1,1})][1 + p(u_{0,0} - u_{1,0})] = 1.$$

Hence

$$(S_l + \mathrm{id}) \ln[1 + p(u_{0,0} - u_{1,0})] = 0,$$

which reduces to the OΔE

$$\ln[1 + p(u_{0,0} - u_{1,0})] = (-1)^l f(k),$$

where $f(k)$ is an arbitrary function.

If $p \neq s \neq 0$ then the substitution $u(k, l) \mapsto (1 - s/p)^k u(k, l) + 1/s$ reduces the NQC equation to

$$\frac{(p^2 - s^2)u_{1,0} - p^2 u_{0,0}}{s^2 u_{1,0}} = \frac{u_{0,1}}{u_{1,1} - u_{0,1}},$$

which can be rearranged as

$$(S_l + \mathrm{id}) \ln\left(1 - \frac{u_{0,0}}{u_{1,0}}\right) = \ln\left(\frac{s^2}{p^2}\right).$$

This may be integrated to yield the OΔE

$$\ln\left(1 - \frac{u_{0,0}}{u_{1,0}}\right) = \frac{1}{2} \ln\left(\frac{s^2}{p^2}\right) + (-1)^l f(k).$$

Finally, if $p = s \neq 0$ then $u(k, l) \mapsto u(k, l) + 1/s$ reduces the NQC equation to

$$(S_l + \mathrm{id})\left(\frac{u_{0,0}}{u_{1,0}}\right) = 2,$$

which yields the OΔE

$$\frac{u_{0,0}}{u_{1,0}} = 1 + (-1)^l f(k).$$

□

We now seek to to simplify the NQC equation by using equivalence transformations. All choices of p, q, r, s are considered, subject only to the two *nondegeneracy constraints*:

(i) $p^2 \neq q^2$;
(ii) at most one element in each of the pairs $\{p, r\}$, $\{p, s\}$, $\{q, r\}$, $\{q, s\}$ is zero.

These constraints will be assumed to hold from here on. We will use point transformations of the form

$$u(k, l) \mapsto \lambda^k \mu^l u(k, l) + f(k, l),$$

where λ, μ are nonzero constants, to simplify the coefficients of the NQC equation.

Case I: $s = r$, $(p^2 - r^2)(q^2 - r^2) \neq 0$

In this case, the NQC equation can be transformed into the *cross-ratio equation*,

$$\frac{\alpha(u_{1,0} - u_{0,0})}{\beta(u_{0,1} - u_{0,0})} = \frac{u_{1,1} - u_{1,0}}{u_{1,1} - u_{0,1}}, \qquad \alpha \neq 0, \quad \beta \neq 0, \quad \alpha \neq \beta. \tag{22}$$

Here $\alpha = p^2 - r^2$ and $\beta = q^2 - r^2$.

If $r = s = 0$, the required transformation is

$$u(k, l) \mapsto u(k, l) + \frac{k}{p} + \frac{l}{q}, \tag{23}$$

which gives $\alpha = p^2$ and $\beta = q^2$. Otherwise, the transformation is

$$u(k, l) \mapsto \left(\frac{p - r}{p + r}\right)^k \left(\frac{q - r}{q + r}\right)^l u(k, l) + \frac{1}{2r}; \tag{24}$$

Case II: $s = r$, $(p^2 - r^2)(q^2 - r^2) = 0$

Here at least one of the coefficients $(p \pm r)$ and $(q \pm r)$ is zero. Furthermore, the nondegeneracy constraints are only satisfied if exactly one such coefficient is zero. Consequently r is nonzero. By using the equivalence transformations generated by Γ_a, we can set $q + r = 0$ without loss of generality; then the NQC equation amounts to

$$\frac{(p + r)u_{1,0} - (p - r)u_{0,0} - 1}{2ru_{0,0} - 1} = \frac{2ru_{1,0} - 1}{(p + r)u_{1,1} - (p - r)u_{0,1} - 1}.$$

The point transformation

$$u(k, l) \mapsto \left(\frac{p - r}{p + r}\right)^k \left(\frac{4r^2}{p^2 - r^2}\right)^l u(k, l) + \frac{1}{2r} \tag{25}$$

reduces the NQC equation to

$$\frac{u_{1,0} - u_{0,0}}{u_{0,0}} = \frac{u_{1,0}}{u_{1,1} - u_{0,1}}. \tag{26}$$

Case III: $s = -r \neq 0$

Here the NQC equation amounts to

$$\frac{(p-r)(u_{1,0}-u_{0,0})-1}{(q-r)(u_{0,1}-u_{0,0})-1} = \frac{(q+r)(u_{1,1}-u_{1,0})-1}{(p+r)(u_{1,1}-u_{0,1})-1}.$$

At least three of the four coefficients must be nonzero, for otherwise the nondegeneracy constraints are violated. We can use Γ_a and Γ_b to set $(p+r)(q+r) \neq 0$. Then the transformation

$$u(k,l) \mapsto u(k,l) + \frac{k}{p+r} + \frac{l}{q+r} \tag{27}$$

simplifies the NQC equation to

$$\frac{(p^2-r^2)(u_{1,0}-u_{0,0})-2r}{(q^2-r^2)(u_{0,1}-u_{0,0})-2r} = \frac{u_{1,1}-u_{1,0}}{u_{1,1}-u_{0,1}}, \qquad p^2 \neq q^2, \quad r \neq 0. \tag{28}$$

Case IV: $s^2 \neq r^2$

First note that the transformation

$$u(k,l) \mapsto u(k,l) + \frac{1}{r+s} \tag{29}$$

reduces the NQC equation to

$$\frac{(p+s)u_{1,0}-(p-r)u_{0,0}}{(q+s)u_{0,1}-(q-r)u_{0,0}} = \frac{(q+r)u_{1,1}-(q-s)u_{1,0}}{(p+r)u_{1,1}-(p-s)u_{0,1}}. \tag{30}$$

At least one of the coefficients in each numerator and denominator in (30) must be nonzero, for otherwise (30) reduces to an OΔE or a trivial identity. Therefore at least four of the eight coefficients are nonzero. Suppose that (30) can be transformed by an element of the group generated by Γ_a, Γ_b into a form for which

$$(p+r)(p-s)(q+r)(q-s) \neq 0. \tag{31}$$

Then the transformation

$$u(k,l) \mapsto \left(\frac{p-s}{p+r}\right)^k \left(\frac{q-s}{q+r}\right)^l u(k,l) \tag{32}$$

reduces (30) to

$$\frac{(p^2-s^2)u_{1,0}-(p^2-r^2)u_{0,0}}{(q^2-s^2)u_{0,1}-(q^2-r^2)u_{0,0}} = \frac{u_{1,1}-u_{1,0}}{u_{1,1}-u_{0,1}}, \qquad p^2 \neq q^2 \quad s^2 \neq r^2. \tag{33}$$

All that remains is to consider whether there are any circumstances in which the coefficients cannot be transformed to satisfy (31). If only one coefficient is zero, it can be transformed into $q+s=0$, which does not violate (31). Suppose then, that $q+s=0$ and that at least one of the factors in (31) is also zero. Clearly, $p-s$ and $q-s$ must be nonzero to satisfy the nondegeneracy constraints. The case $q+r=0$ does not need to be considered, as this violates $s^2 \neq r^2$. Finally, if $p+r=q+s=0$ then the rotation Γ_a transforms these conditions to $q+s=-p+r=0$, and so (31) is satisfied.

This completes the classification of the simplified forms of the NQC equation. We have shown that, up to equivalence transformations, the only nondegenerate cases of NQC may be mapped to one of (22), (26), (28) and (33).

4. Three-point conservation laws of the simplified equations

In this section we present a complete classification of the three-point conservation laws of (22), (26), (28) and (33). These conservation laws have been obtained by the method described in section 2. The computer algebra system MAPLE was used to carry out the calculations, details of which are omitted.

Case I. A complete set of three-point conservation laws for the cross-ratio equation (22) is

(i) $F = \dfrac{\alpha}{u_{0,1} - u_{0,0}}, \qquad G = -\dfrac{\beta}{u_{1,0} - u_{0,0}},$

(ii) $F = \dfrac{\alpha u_{0,0}}{u_{0,1} - u_{0,0}}, \qquad G = -\dfrac{\beta u_{1,0}}{u_{1,0} - u_{0,0}},$

(iii) $F = \dfrac{\alpha u_{0,0} u_{0,1}}{u_{0,1} - u_{0,0}}, \qquad G = -\dfrac{\beta u_{0,0} u_{1,0}}{u_{1,0} - u_{0,0}},$

(iv) $F = (-1)^{k+l}(2\ln[u_{0,1} - u_{0,0}] - \ln[\alpha]), \qquad G = -(-1)^{k+l}(2\ln[u_{1,0} - u_{0,0}] - \ln[\beta]).$

These conservation laws also hold when $\alpha = \alpha(l)$ and $\beta = \beta(k)$.

Case II. A complete set of three-point conservation laws for equation (26) is

(1) $F = \dfrac{u_{0,1}}{u_{0,0}} \qquad G = \dfrac{u_{1,0}}{u_{1,0} - u_{0,0}},$

(2) $F = (-1)^{k+l}\ln[u_{0,0}] \qquad G = -(-1)^{k+l}\ln[u_{1,0} - u_{0,0}].$

Case III. A complete set of three-point conservation laws for equation (28) is

(1) $F = \ln\left[\dfrac{(q^2 - r^2)(u_{0,1} - u_{0,0}) - 2r}{u_{0,1} - u_{0,0}}\right] \qquad G = -\ln\left[\dfrac{(p^2 - r^2)(u_{1,0} - u_{0,0}) - 2r}{u_{1,0} - u_{0,0}}\right],$

(2) $F = (-1)^{k+l}\ln[(u_{0,1} - u_{0,0})((q^2 - r^2)(u_{0,1} - u_{0,0}) - 2r)]$
$G = -(-1)^{k+l}\ln[(u_{1,0} - u_{0,0})((p^2 - r^2)(u_{1,0} - u_{0,0}) - 2r)].$

Case IV. A complete set of three-point conservation laws for equation (33), except for the cases $p = r, q = -s$ and $p = -s, q = r$, is

(1) $F = \ln\left[\dfrac{u_{0,1} - u_{0,0}}{(q^2 - s^2)u_{0,1} - (q^2 - r^2)u_{0,0}}\right]$
$G = -\ln\left[\dfrac{u_{1,0} - u_{0,0}}{(p^2 - s^2)u_{1,0} - (p^2 - r^2)u_{0,0}}\right],$

(2) $F = (-1)^{k+l}\ln[(u_{0,1} - u_{0,0})((q^2 - s^2)u_{0,1} - (q^2 - r^2)u_{0,0})]$
$G = -(-1)^{k+l}\ln[(u_{1,0} - u_{0,0})((p^2 - s^2)u_{1,0} - (p^2 - r^2)u_{0,0})].$

If $p = r, q = -s$ in (33) then the three-point conservation laws are

(1) $F = \ln\left[\dfrac{u_{0,1} - u_{0,0}}{u_{0,0}}\right] \qquad G = -\ln\left[\dfrac{u_{1,0} - u_{0,0}}{u_{1,0}}\right],$

(2) $F = (-1)^{k+l}\ln[u_{0,0}(u_{0,1} - u_{0,0})] \qquad G = -(-1)^{k+l}\ln[u_{1,0}(u_{1,0} - u_{0,0})],$

(3) $F = (k + l)\ln\left[\dfrac{u_{0,1} - u_{0,0}}{u_{0,1}}\right] - \ln[u_{0,1}u_{0,0}]$
$G = -(k + l)\ln\left[\dfrac{u_{0,0}(u_{1,0} - u_{0,0})}{u_{1,0}{}^2}\right].$

Finally, if $p = -s, q = r$ then the three-point conservation laws are

(1) $F = \ln\left[\dfrac{u_{0,1} - u_{0,0}}{u_{0,1}}\right] \qquad G = -\ln\left[\dfrac{u_{1,0} - u_{0,0}}{u_{0,0}}\right],$

(2) $F = (-1)^{k+l} \ln[u_{0,1}(u_{0,1} - u_{0,0})] \qquad G = -(-1)^{k+l} \ln[u_{0,0}(u_{1,0} - u_{0,0})],$

(3) $F = (k+l)\ln\left[\dfrac{u_{0,0}(u_{0,1} - u_{0,0})}{u_{0,1}^2}\right] \qquad G = -(k+l)\ln\left[\dfrac{u_{1,0} - u_{0,0}}{u_{1,0}}\right] + \ln[u_{1,0}u_{0,0}],$

5. Constructing the conservation laws of the NQC equation

Having classified the conservation laws of the simplified representatives of each equivalence class, we now show how to obtain conservation laws of the original NQC equation for a particular example. This demonstrates the method of constructing conservation laws for any nondegenerate choice of coefficients. Suppose that

$$(p, q, r, s) = (-1, 2, 1, 1); \tag{34}$$

then the NQC equation amounts to

$$\frac{2u_{0,0} - 1}{3u_{0,1} - u_{0,0} - 1} = \frac{3u_{1,1} - u_{1,0} - 1}{2u_{0,1} - 1}. \tag{35}$$

This case satisfies the condition $r = s$, $p + r = 0$, so (35) is a subcase of Case II. From the discussion in section 3, all equations in Case II can be obtained by applying one of the rotations generated by Γ_a to

$$\frac{(\hat{p} + \hat{r})\hat{u}_{1,0} - (\hat{p} - \hat{r})\hat{u}_{0,0} - 1}{2\hat{r}\hat{u}_{0,0} - 1} = \frac{2\hat{r}\hat{u}_{1,0} - 1}{(\hat{p} + \hat{r})\hat{u}_{1,1} - (\hat{p} - \hat{r})\hat{u}_{0,1} - 1}. \tag{36}$$

To obtain the NQC equation with coefficients (34) we need to apply Γ_a to (36) once. Therefore, from (14), the choice of coefficients in (36) that corresponds to (35) is

$$\frac{3\hat{u}_{0,0} - \hat{u}_{1,0} - 1}{2\hat{u}_{0,0} - 1} = \frac{2\hat{u}_{1,0} - 1}{3\hat{u}_{0,1} - \hat{u}_{1,1} - 1}. \tag{37}$$

Equation (37) has two conservation laws which can be obtained from the conservation laws for (26) by inverting the transformation (25). After rescaling (for convenience), these are

(1) $\hat{F}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{0,1}) = -\dfrac{3(2\hat{u}_{0,1} - 1)}{2(2\hat{u}_{0,0} - 1)} \qquad \hat{G}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{1,0}) = \dfrac{2\hat{u}_{1,0} - 1}{3\hat{u}_{0,0} - \hat{u}_{1,0} - 1},$

(2) $\hat{F}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{0,1}) = (-1)^{\hat{k}+\hat{l}+1} \ln[1 - 2\hat{u}_{0,0}] \qquad \hat{G}(\hat{k}, \hat{l}, \hat{u}_{0,0}, \hat{u}_{1,0}) = (-1)^{\hat{k}+\hat{l}} \ln[3\hat{u}_{0,0} - \hat{u}_{1,0} - 1].$

So (17) tells us that the NQC equation with $(p, q, r, s) = (-1, 2, 1, 1)$ has the two conservation laws

(1) $F(k, l, u_{0,0}, u_{0,1}) = \hat{G}(-l - 1, k, u_{0,1}, u_{0,0}) = \dfrac{2u_{0,0} - 1}{3u_{0,1} - u_{0,0} - 1}$

$G(k, l, u_{0,0}, u_{1,0}) = -\hat{F}(-l, k, u_{0,0}, u_{1,0}) = \dfrac{3(2u_{1,0} - 1)}{2(2u_{0,0} - 1)},$

(2) $F(k, l, u_{0,0}, u_{0,1}) = (-1)^{k-l} \ln[3u_{0,1} - u_{0,0} - 1]$

$G(k, l, u_{0,0}, u_{1,0}) = -(-1)^{k-l} \ln[1 - 2u_{0,0}].$

6. Conclusion

The classification of three-point conservation laws for all nondegenerate cases of the NQC equation has been completed. The number and form of such conservation laws depends upon the parameter values, even though the NQC equation is integrable for all nondegenerate parameter choices. Commonly, this type of behaviour is a consequence of the appearance of extra symmetries when parameters take particular values. In principle, it would be possible to extend the classification to higher-order conservation laws, as we have done previously for the dKdV equation [22]. However, to do this by the direct method for the NQC equation would involve massive calculations.

It is interesting to note that although each of the four simplified forms of the NQC equation is integrable, it is hard to check this by using consistency-on-a-cube [1], because we have eliminated parameters that correspond to the edges of the quad-graph. Indeed, some of the simplified equations, such as (26), are not included in [1] because they do not admit symmetric properties. Note that (26) does not admit a three-leg form representation; however, its solutions satisfy

$$\frac{u_{1,1}}{u_{0,0}} + \frac{u_{0,0}}{u_{1,-1}} - \frac{u_{0,0}}{u_{-1,-1}} - \frac{u_{-1,1}}{u_{0,0}} = 0,$$

which is similar to the Toda system that can be constructed from quad-graph equations that have a three-leg form.

It remains to be seen whether the simplified equations that are not in the Adler, Bobenko and Suris classification [1] can be linearized by a non-point transformation in the manner of Hietarinta's example [7, 21].

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